EXISTENCE THEOREM FOR THE PRICES FIXED POINT PROBLEM OF THE OVERLAPPING GENERATIONS MODEL, VIA METRIC SPACES ENDOWED WITH A GRAPH

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Abstract

The aim of this paper is to study the existence of the solution for the overlapping generations model, using fixed point theorems in metric spaces endowed with a graph. The overlapping generations model has been introduced and developed by Maurice Allais (1947), Paul Samuelson (1958), Peter Diamond (1965) and so on. The present paper treats the case presented by Edmond (2008) in (Edmond, 2008) for a continuous time. The theorem of existence of the solution for the prices fixed point problem derived from the overlapping generations model gives an approximation of the solution via the graph theory. The tools employed in this study are based on applications of the Jachymski fixed point theorem on metric spaces endowed with a graph (Jachymski, 2008).

Key words: the overlapping generations model, fixed points theorems, graph, Banach G-contraction, Urysohn integral equations

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1. Introduction

Fixed points theorems play an important role in economics because they provide the existence and uniqueness of the solution for economics existence problems. This paper deals with the existence of the equilibrium price function for the fixed point problem $p = T_{\sigma} p$, in the case of $\sigma \to 1$, which models the overlapping generations problem. The equilibrium fixed point problem is represented as a nonlinear Urysohn integral equation which needs to be solved for an unknown price function *p* (Edmond, 2008). The method for obtaining the approximations to equilibrium prices used in this paper and also in (Edmond, 2008) is developed for the special cases of integral linear equations.

This paper proposes a way of proving the existence of the price solutions using a certain constructed graph *G* and obtaining the approximants to equilibrium prices for the general case of nonlinear Urysohn integral equation of Fredholm type. A similar approach is treated in (Bojor, et al., 2013).

2. The overlapping genrations model – general aspects

The overlapping generations models (OLG) are well known in the macroeconomic literature. Since 1947 when M. Allais[1], P. Samuelson[2] (1958) and P. Diamond[3]

[1] Maurice Allains (1911-2010) – French economist, winner of the Nobel Memorial Prize in Economics in 1988

(1965) introduced and developed the OLG model, notable improvements and different versions were published in many researches (see (Cass, et al., 1967), (Demichelis, et al.), (Ríos-Rull, 1996), (Weil, 1989), (d'Albis, et al., 2007)).
The overlapping generation model is an economic model which studies the behavior of the generations and the interactions between them. In this model, the generations are represented by the economic agents. Every generation lives a finite length of time long enough to live one period at least with the next generation of agents.
Starting from the phrase "Saving today for tomorrow" (the young generation contributes for the older generations) the OLG model tried to give answers to the questions "How much is enough to help?" and "What is the benefits level that the beneficiary generation should receive?" Thus, *the purpose of the OLG model is to identify the relation between savings and consumption, to find the solutions for financial resources for the generation that no longer produces, to study the consumer behavior changes in different phases of their life* (Voineagu, et al., 2012).
The development of the OLG models caught the attention of many researchers who took into consideration the demographic evolutions, the endowments of the generation, the preferences and budget constraints, the optimization of the model and multiple equilibrium such as the consumption-saving equilibrium, the consumption-price equilibrium and so on.
The latest interest (starting with Blanchard (Blanchard, 1985)) in improving the OLG models includes the continuous time where agents live for a given finite interval of time so that the life-cycle behavior is possible. The life-cycle hypothesis ensures the stability of the agent's lifestyle: "to keep their consumption levels approximately the same in every period." (Wikipedia, 2014). Moreover, the continuous time overlapping generations model "simultaneously permits agents to have many decisions per lifetime and is computationally simple" (Edmond, 2008). Also, taking finite lifetime into account, the model offers "a realistic approach to the study of macroeconomic effects of government budget deficits and government debt." (Groth, 2011)
The results of this paper are generated by Edmond for equilibrium prices of the continuous time OLG model.

3. Mathematical general results

3.1. Mathematical-economics model

This section briefly describes the continuous time overlapping generations model with a finite-horizon life-cycle (for details see (Edmond, 2008)).

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[2] Paul Samuelson (1915-2009) – American economist, winner of the Nobel Prize in Economics in 1970
[3] Peter Diamond (1940) – American economist, winner of the Nobel Prize in Economics in 2010

The model considers a deterministic exchange economy populated by an infinity of overlapping generations of agents. The time is continuous and it is denoted by $t\in[0,\infty)$.

From the *demographic* point of view, the interval $\mathcal{G}(t)=(t-l,t]$ denotes the set of all generations alive at date ≥ 0 , where l is the life period of agents. Also, the set $\mathcal{A}(v)=\{t\geq 0 \mid \max\{0,v\}\leq t<v+l\}$ denotes the set of dates over which generation $v\in(-l,\infty)$ lives.

From the *endowments* point of view, $y(t,v)$ is the amount of the endowment of a generation v at the time t. Let $Y(t)=\int_{\mathcal{G}(t)} y(t,v)dv>0$ denote the aggregate endowment at the time $t\geq 0$. The report $y(t,v)/Y(t)>0$ gives the density of the endowment of the generation v . The net assets of generation v at time t is denoted with (t,v).

Each individual has *preferences* over dated consumption goods represented by the function

(1) $$\int_{\mathcal{A}(v)} e^{-\rho(t-v)}u[c(t,v)]dt,\quad u(c)=\frac{c^{1-\sigma}-1}{1-\sigma},\ \rho\geq 0,\ \sigma>0$$

The intertemporal individual wealth, as of given prices p of consumption at the date $t=0$, can be written:

$$W(p,v)=\int_0^\infty p(s)y(s,v)ds+p(0)a(0,v).$$

The *optimization* problem means that each individual chooses consumption $c(t,v)\geq 0$ to maximize utility (1) taking into account their intertemporal constraint $\int_{\mathcal{A}(v)} p(t)c(t,v)dt=\int_{\mathcal{A}(v)} p(t)y(t,v)dt+p(v)a(v,v)$. This problem conduces to the consumption function

$$c_\sigma(p,t,v)=\frac{\alpha_\sigma(p,t,v)(W(p,v))}{p(t)},$$

where $\alpha_\sigma(p,t,v)$ denotes the expenditure shares:

$$\alpha_\sigma(p,t,v)=\frac{e^{-\frac{\rho}{\sigma}t}p(t)^{\frac{\sigma-1}{\sigma}}}{\int_{\mathcal{A}(v)} e^{-\frac{\rho}{\sigma}s}p(s)^{\frac{\sigma-1}{\sigma}}ds}.$$

The model tries to establishes the *equilibrium* between consumption c and the price p . The equilibrium is feasible if $c\geq 0$, $p>0$, each individual chooses consumption to maximize utility and markets clear $\int_{\mathcal{G}(t)} c_\sigma(p,t,v)dv=\int_{\mathcal{G}(t)} y(t,v)dv=:Y(t)$. This means that it is necessary “to find a function p that ensures that markets clear on all dates.” (Edmond, 2008). Edmond rewrites the market clearing condition as a second kind integral equation (not as the first kind as it was expressed before) for which the unknown price function can be numerical found more easily than it can be found in the case of the first kind integral equations.

Thus, the integral equation representation of equilibrium price is the nonlinear integral equation

$$p(t) = \int_0^\infty p(s) k_\sigma(p,t,s) ds + f_\sigma(p,t) \quad (2),$$

where

$$k_\sigma(p,t,s) = \int_{S(t)} \frac{\alpha_\sigma(p,t,v)(y(s,v))}{Y(t)} dv$$

and

$$f_\sigma(p,t) = \int_{S(t)} \frac{\alpha_\sigma(p,t,v)(a(0,v))}{Y(t)} dv.$$

The equation (2) is a *nonlinear Urysohn integral equation*.

Proposition 1. ((Edmond, 2008), p. 600) *Intertemporal prices p solve the nonlinear integral equation (2).*

- ((Edmond, 2008), p. 600) *The prices p solve a fixed point problem of the form*

$$p = T_\sigma p \quad (3)$$

where T_σ is a nonlinear integral operator T_σ that take prices as an argument: $(T_\sigma p)(t) = \int_0^\infty k_\sigma(p,t,s) ds + f_\sigma(p,t)$ *and it represents the right hand side of (2).*

In the main section of this paper, Section 3, we shall give a different demonstration of existence of the equilibrium prices for the definite integral linear Urysohn equation. We shall use recently results of Jachymski fixed points theory ((Jachymski, 2008)).

3.2. Theoretical frame

Jachymski’s frame uses the general notions of: metric spaces, Picard operator, graph and Bancah G-connected graph. These notions are presented in this section.

Definition 1. *Let X be a set and $d: X \times X \to \mathbb{R}$ a function. The function d is called metric on X if d satisfies the following conditions:*

i. $d(x,y) \geq 0$, $\forall x,y \in X$ *; (non-negative)*

ii. $d(x,y) = 0 \Leftrightarrow x = y$ *;*

iii. $d(x,y) = d(y,x)$, $\forall x,y \in X$ *; (symmetry)*

iv. $d(x,z) \leq d(x,y) + d(y,z)$*. (triangle inequality)*

Let T be a selfmap of a metric space (X,d), i.e. $T: X \to X$ is an operator.

Definition 2. *Let X be a nonempty set and $T: X \to X$ a selfmap. An element $x \in X$ is called fixed point for T if $T(x) = x$.*

Definition 3. (Petruşel, et al., 2006) *T is a Picard operator if T has a unique fixed point x^* and $\lim_{n\to\infty} T^n x = x^*$, $\forall\, x \in X$.*

It is known that the components of a graph are vertices (or nodes) and edges.

In Jachymski results, G is directed graph with no parallel edges such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops of the diagonal of the Cartesian product (denoted with Δ) $X \times X$, i.e. $E(G) \supseteq \Delta$.

Definition 4. (Johnsonbaugh, 1997) *Let G be a graph. If $x, y \in V(G)$, then a path in G from x to y of length N ($N \in \mathbb{N}$) is a sequence $(x_i)_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \ldots, N$.*

Definition 5. (Johnsonbaugh, 1997) *A graph G is connected if there is a path between any two vertices.*

Jachymski studied the conditions derived from the graph theory for which a selfmap T of a metric space is a Picard operator. These conditions are presented in Theorem 2

Definition 6. (Jachymski, 2008) *We say that a mapping $T: X \to X$ is a Banach G-contraction if T preserves edges of G, i.e.*

$$\forall x, y \in X, \big((x, y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G)\big) \tag{4}$$

and T decreases weights of edges of G in the following way:

$$\exists \alpha \in (0,1), \forall x, y \in X,$$

$$\big((x, y) \in E(G) \Rightarrow d(T(x), T(y)) \le \alpha \cdot d(x, y)\big). \tag{5}$$

Theorem 1. (Jachymski, 2008) *Let (X, d) be complete and let the triple (X, d, G) have the following property: for any $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$.*

Let $T: X \to X$ be a Banach G-contraction and $X_T = \left\{x \in \frac{X}{x, Tx} \in E(G)\right\}$.
Then the following statements hold:

1. Card $\operatorname{card} \operatorname{Fix} T = \operatorname{card}\left\{\frac{[x]_{\tilde{G}}}{x} \in X_T\right\}$.
2. $\operatorname{Fix} T \neq \emptyset$ *iff* $X_T \neq \emptyset$.
3. T *has a unique fixed point iff there exists* $x_0 \in X_T$ *such that* $X_T \subseteq [x]_{\tilde{G}}$.

4. *For any* $x \in X_T$, $T\big|_{[x]_{\tilde{G}}}$ *is a Picard operator.*
5. *If* $X_T \neq \emptyset$ *and* G *is weakly connected, then* T *is a Picard operator.*
6. *If* $X^* = \cup \left\{ \frac{[x]_{\tilde{G}}}{x} \in G \right\}$ *then* $T\big|_{X}^{*}$ *is a weakly Picard operator.*
7. *If* $T \subseteq E(G)$ *then* T *is a weakly Picard operator.*

4. Mathematical main result

We shall prove an existence theorem of the solutions of the linear Urysohn integral equation of Fredholm type via graph theory. The existence and uniqueness of the solution of Fredholm integral equation type, in metric spaces endowed with a graph, was studied in (Bojor, et al., 2013). Because the OLG model does not respect the G-contraction property, the equilibrium functions are not unique.
The linear integral equation rises in the practical case of log utility. The linear Urysohn integral equation (2) in the particular bounded case is

$$p(t) = \int_0^a p(s)k(t,s)ds + f(t), \quad a \in \mathbb{R}_+^* \tag{6}$$

This is a linear Fredholm equation of the second kind. Due to Fredholm theory,

(i) the function $k(t,s)$ is known as the kernel of the integral equation; it must be continuous on the domain $D = \{(t,s) \mid 0 \le t,s \le a\}$;

(ii) the function $f(t)$ is called the free term of the integral equation and it is a continuous function on the interval $[0,a]$.

These hypotheses are used in generating the approximations of the solution p of the integral Fredholm equations.
Throughout this section we assume that (X,d) is a metric space.
We introduce the following relation of partial order defined on the set of continuous functions:
$f,g \in X, f \le g \Leftrightarrow f(x) \le g(x), \forall x \in [0,a]$

Definition 7. *Let the set* $X = C[0,a]$ *be the set of all continuous functions on* $[0,a]$ *endowed with the metric*

$$d(p_1,p_2) = \max_{t\in[0,a]} |p_1(t) - p_2(t)|, \quad \forall p_1,p_2 \in X.$$

In this hypothesis we define the graph $G \subset X \times X$ *by*

$$p_1,p_2 \in E(G) \Leftrightarrow p_1 \le p_2.$$

Theorem 2. *(Existence Theorem) If the conditions (i), (ii) holds true and we suppose that*

(iii) f *is a positive function, i.e.* $f(t) \ge 0, \forall t \in [0,a]$

(iv) $\exists \alpha = a \cdot M \in (0,1)$*, where* $M = \max_{(t,s)\in D} |k(t,s)|$, $a \in \mathbb{R}_+^*$,

then the linear Urysohn integral equation of Fredholm type of second kind (6) has at least one equilibrium price solution.

Proof. We define a map T by $Tp = \int_0^a p(s)k(t,s)ds + f(t)$, $\forall t,s \in [0,a]$. Because $\forall p \in X \Rightarrow Tp \in X$. It follows that T is a selfmap. Moreover, because the kernel k is a continuous function on D, then k is bounded. Let $M = \max_{(t,s)\in D} |k(t,s)|$ be the upper bound.

We identify the metric space (X,d) from Jachymski's theorem with $X = C[0,a]$, respectively $d(p_1,p_2) = \max_{t\in[0,a]} |p_1(t) - p_2(t)|$ for all $p_1,p_2 \in X$. Let this metric space be endowed with the graph G defined in Definition 7.

Now we prove that the map T is a G-contraction using Definition 6, i.e. T preserves the edges of G and T decreases the weights of the edges of G.

First condition means $(p_1,p_2) \in E(G) \Rightarrow (Tp_1,Tp_2) \in E(G)$. Indeed, let $p_1,p_2 \in E(G)$. It follows that $p_1(t) \le p_2(t)$. Thus

$$(7)\quad (Tp_1)(t) = \int_0^a p_1(s)k(t,s)ds + f(t) \le \int_0^a p_2(s)k(t,s)ds + f(t) = (Tp_2)(t),\ \forall t \in [0,a].$$

The second condition is true because $\exists \alpha = aM \in (0,1)$ such that $((p_1,p_2) \in E(G) \Rightarrow d(Tp_1,Tp_2) \le \alpha \cdot d(p_1,p_2))$. Indeed, let $p_1,p_2 \in E(G)$. It follows that $p_1(t) \le p_2(t)$. Thus

$$|Tp_1 - Tp_2| \le \int_0^a |p_1(s) - p_2(s)| \cdot |k(t,s)|ds \Rightarrow$$

$$(8)\quad d(Tp_1,Tp_2) \le s \cdot d(p_1,p_2)$$

From (7), (8) it follows that T is a G-contraction.

We are in the hypothesis of Jachymski's theorem (Theorem 1) and we apply the result no. 4 to prove that the map T is a Picard operator. We know that the set $X_T = \left\{p \in \frac{X}{p,Tp} \in E(G)\right\}$ is nonempty because there exists an initial price function $p \equiv 0$ such that $p \equiv 0 \le \int_0^a 0 + f = T0 \equiv Tp$, f being a positive function on $[0,a]$. Thus, $\forall p \in X_T$, the restriction of the map T to the equivalence class $[p]_{\tilde{G}}$ is a Picard operator. The notion $[p]_{\tilde{G}}$ is used in (Johnsonbaugh, 1997) as the equivalence class of the relation R defined on $V(G)$ by the rule: $xRy\ if\ there\ is\ a\ path\ in\ \tilde{G}\ from\ x\ to\ y$, where $\tilde{G}$ denotes the undirected graph obtained from G by ignoring the directions of edges.

This proves that for any equivalence class there exists one fixed point $p^* \in C[0,a]$ such that $Tp^* = p^*$ which is a solution for the integral equation (6). ■

Fixed point theorems usually provide a method for construction of the solution by iterations. Accordingly (Berinde, 2007),

Proposition 1. *A method for construction of the solution* $\{[p]_n\}$ *is*

$p_{n+1}(t)=\int_0^a k(t,s)p_n(s)ds+f(t)=T^n p$, $\forall t\in[0,a], n=0,1,2,\dots$. *and* $p_0(t)=0$, $\forall t\in[0,a]$.

The error estimates is $d(T^n p,p^*)\le\frac{(aM)^n}{1-aM}\cdot d(0,T0), n=0,1,2,\dots$ *where* p^* *is the fixed point solution.*

Conclusion

We apply a fixed point theorem via graph theory to obtain existence theorem for the linear Urysohn integral equation on $[0,a]$. The main tools employed in this paper are based on Jachymski fixed point theory on metric spaces endowed with a graph. The aim of this paper was to investigate the applicability of these mathematical results on certain economical problem, the equilibrium of the continuous overlapping generations model with a finite-horizon life-cycle. In the case of log utility, the equilibrium reduces to a linear integral equation - Urysohn integral equation of Fredholm type. Thus, we define a graph G that consists on price-functions from the continuous space $C[0,a]$ with property that any two price functions form an edged of the graph if and only if there exits the partial order relation $p_1\le p_2$. We also consider the Urysohn operator T (Edmond sows that this operator reduces the integral equation to a fixed point equation $Tp=p$) as a selfmap defined on a metric space and we prove that it is a G-contraction. Further, applying a property from Jacyimski theorem, we conclude that for any equivalence class for which all the price-functions are connected to each other, the fixed point problem has at least one solution. The solutions may be obtained by iterating the liner Urysohn integral equation and the error estimate is given.

Further investigates may consist on finding the rate of convergence and stability of the above results, or to give the existence theorem for the nonlinear Uryshon integral equation on $[0,\infty)$.

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