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VARIOUS CONVEXITIES AND SOME RELEVANT PROPERTIES OF CONSUMER PREFERENCE RELATIONS

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Abstract: The concept of convexity plays an important role in the study of economics and consumer theory. For the most part, such studies have been conducted on the assumption that consumer preferences are a binary relation that is complete, reflexive and transitive on the set X of consumption choices. However, each consumer is a biological being with multidimensional physiological needs so that possible consumptions from different dimensions cannot be compared by using preferences. By removing that unrealistic assumption, this paper examines how the various concepts of convex preferences and relevant properties can be re-established. We derive a series of 10 formal propositions and construct 6 examples to show that (a) a weighted combination of two possible consumptions is not necessarily comparable with any of the consumptions; (b) not every convergent sequence of a consumer's preferred consumptions asymptotically preserves that consumer's

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preference preordering; (c) not all preferences satisfy either positive multiplicativity or additive conservation; (d) three types of preference convexities – weak convexity, convexity and strong convexity – can all be introduced into general convex spaces. This paper concludes with some research topics of expected significance for future works.

Keywords: additive conservation; convex space; Euclidean space; half-space; positive multiplicativity; possible consumption; preorder; utility function.

JEL codes: D11.

1. Introduction

As a physiological being or a form of life, each consumer, be it an individual person or a business firm, has to first satisfy his basic needs of survival before considering the consumption of more luxurious goods and services. For basic survival, the need of each living being has to be met multidimensionally, such as the dimension of shelter or office domain, that of various nutrition or business supplies, etc. Examples of such multidimensionality for survival appear frequently in different levels of living. For instance, tickets to different world series games, soft drink choices, or alternatives of hotels represent consumption from three different dimensions, they cannot be directly compared by using a consumer's preferences. In other words, a consumer cannot compare consumption alternatives in one dimension (light clothing vs heavy clothing) with ones from another dimension (food A vs food B) by simply applying his preferences.

In other words, the set of a consumer's possible consumptions is not completely ordered by his preferences. Although such incompleteness of consumer preferences was noticed by Ok (2002), he only considered it from the angle of bounded rationality and consumers' indecisiveness. Ok cited Aumann (1962), Bewley (1986) and Mandler (1999) without noticing the multidimensionality of a consumer's physiological needs. By contrast, this paper employs the concept of multidimensionality to ask and answer two basic questions: First, under what conditions will the most basic properties of consumer preferences that have been derived from the assumption of complete preferences still hold true when preferences are incomplete? Second, under an incomplete-preference scenario, what will the three concepts of convexities of consumer preferences – weak convexity, convexity and strong convexity – look like?

These problems are undoubtedly very important both theoretically and practically. On the theoretical front, the completeness of consumer preferences is one of the most fundamental assumptions in investigations of consumer behaviors and consumer decision-making (e.g., Debreu, 1959; Hervés-Beloso & Cruces, 2019; Mas-Collel et al., 1995). With such a key assumption in place, various preference convexities have been subsequently introduced to capture different aspects of consumer behaviors and characteristics (e.g., Brandl & Brandt, 2020; Debreu, 1959; Mas-Collel et al., 1995). These concepts also appear in other fields of study and are considered canonical (e.g., Chateauneuf & Tallon, 2002). In particular, these convex preferences have been employed to express how each consumer may be inclined toward diversification and how consumers tend to prefer more diverse bundles of products, goods, and services to extremely uniform ones. The key implication of these studies for this article is that when the canonical assumption of completeness is altered, all the previously established conclusions need to be reexamined for their validity.

In other words, the earlier assumed completeness of consumer preferences laid the foundation for various economic theories to appear. However, when that completeness assumption is replaced by an incompleteness assumption, all relevant conclusions of the subsequent economics will most likely take correspondingly different forms. Those forms are what this paper strives to establish.

On the practical front, the literature has already suggested the need for such reexamination. It has been widely noted that the existing economic theories become useless when they are employed to provide guidelines for practical purposes or to make predictions about what drastic changes might occur in the economy. For relevant details, see Paul Krugman (New York Times, 2009-09-02) and Paul De Grauwe (Financial Times, 2009-07-21). By replacing an unrealistic assumption with one that is more relevant to life, however, one can expect to produce consequent theories that will be more useful in practice than the older, prevalent theories have been.

This paper, therefore, enriches economic literature through that replacement and by showing the following unorthodox conclusions: (1) The weighted combination of two possible consumptions is generally not comparable with any of the possibilities; (2) Not every convergent sequence of preferable consumptions of a consumer asymptotically preserves that consumer's preference preordering; (3) Not all preferences satisfy either positive multiplicativity or additive conservation; (4) all three types of preference convexities – weak convexity, convexity, and strong convexity – can all be introduced into general convex spaces.

To develop those conclusions, the following Section 2 provides a literature review. Section 3 then lays out the basics of a consumer and provides the necessary terminology for the rest of the presentation. Section 4 looks at various types of convex preferences, concepts of asymptotically preserving preferences, and conditions of positive multiplicativity and additive conservation. The section further provides a study on how to develop the concepts of weak-convex, convex and strong-convex preferences in a general convex space. Finally, Section 5 summarizes and concludes the paper.

2. Literature Review

Relevant literature to this work consists of two parts: (1) studies on incomplete preference relation, and (2) studies on convex preferences.

Regarding the incomplete preferences, Dubra and Ok (2002) and Ok (2002) are the first to recognize the importance of imposing the condition of incompleteness on a consumer's set X of possible consumptions. In particular, X is said to be incomplete, if there are possible consumptions $x, y \in X$ such that x is not preferred over y, and y is also not preferred over x. More specifically, Dubra and Ok (2002) develop a risky-choice model, where the considered consumer naturally has such a preference relation that cannot compare each and every pair of possible consumptions. Ok (2002) looks at how to represent an incomplete preference relation by means of a vector-valued utility function, where the conventional idea of a real-number valued utility representation is no longer possible.

Based on these pioneering works, a good number of scholars have looked at many related but different issues addressed in the prevalent consumer theory. For example, Alonso et al. (2010) develop a web-based consensus support system for decision-makers who possess incomplete preference relations. Meng and Chen (2015) introduce a group decision-making method to deal with incomplete preference information. Cettolin and Riedl (2019) test for complete and/or incomplete preferences by designing and conducting particular experiments. These representative works indicate the rise of an important research area.

To enrich this literature, we have gone beyond the known justification for studying incomplete preference relations. For example, Ok (2002) only considers it in terms of bounded rationality and consumers' indecisiveness, factors which had previously been noticed by Aumann (1962), Bewley (1986) and Mandler (1999). By contrary, we note in the current paper that the root of all human consumption lies in physiological or business needs for survival; and only when survival is no longer a concern will consumers look further to the consumption of more luxurious products, goods and services. However, even at the level of survival, the needs of a living being are multidimensional, including such dimensions as the need for shelter, the need for food, the need for drinks, the need for medicine, etc. When two commodities from two different dimensions required for survival are presented, no consumer can really say which commodity is preferred over another, because both commodities are needed for basic survival.

We now turn to the second part of our literature review, which focuses on studies in convex consumer preferences. Extensive research has been published in both concepts of various convexities and applications of such concepts through the investigations of diverse topics. For purposes of our current paper, it is sufficient to note that previous authors have used that extensive research to conclude that convexity represents one of the canonical concepts in economic theories (Jehle &

Reny, 2000; Mas-Collel et al., 1995; Silberger, 2000; Simon, & Blume, 1994). Furthermore, convexity is both a convenience for analytical reasoning and an intuitive property about the preferences of consumers. Intuitively, if consumers have no real preferences, then convexity reflects a key algebraic characteristic of the consumption preferences of consumers: consumers incline more towards a balanced diversification of commodity consumptions, rather than toward any extreme composition.

The contribution of this work to the existent literature is that we reexamine some of the key properties of the three types of convex preferences – weak convexity, convexity and strong convexity. By doing so, we are able to provide their corresponding general forms for the case that a consumer's set of possible consumptions is not completely ordered by his preference relation.

For the reader to go through the following reasoning smoothly, let us define the order relation $\leq$ on $\mathbb{R}^\ell$ as follows, where $\mathbb{R}$ represents the set of all real numbers and $\ell \in \mathbb{N}$ (= the set of all natural numbers). For $x^1 = (x_1^1, x_2^1, \dots, x_\ell^1)$ and $x^2 = (x_1^2, x_2^2, \dots, x_\ell^2) \in \mathbb{R}^\ell$,

$$x^1 \leq x^2 \text{ if and only if } x_h^1 \leq x_h^2, \text{for each } h = 1,2, \dots, \ell.$$

For the convenience of communication, the following section lays down the elementary setup for the technical reasonings of the rest of this paper.

3. The Basics of a Consumer

A consumer decides what bundle of commodities to consume. Such a bundle is known as a consumption plan (or consumption), as is in the literature (Debreu, 1959; Levin & Milgrom, 2004; Mas-Collel et al., 1995). And without loss of generality, the concept of time is ignored here. In other words, commodities are chosen now for both the current time and for the future. The consumer identifies the amounts of all commodities that he will consume and/or provide within a set of constraints, such as whether those commodities needed for survival. At the same time, the total cost of any chosen consumption cannot go beyond his level of wealth.

Assume that the market place contains m consumers, for some $m \in \mathbb{N}$. For consumer i (= 1, 2, …, m), the quantities of his commodity inputs (i.e., those products, goods and services consumed by i) are written as positive numbers, while the quantities of his commodity outputs (= what i offers to the world) negative numbers. Assume that the set of all commodities, consisting of a total of ℓ different kinds, are ordered and named as $h =$ 1, 2, …, ℓ. As commonly done in economic analysis (e.g., Pancs, 2018), assume that the quantity of each commodity, as shown in a consumption plan, is a real number.

In this model setup, assumed include (i) perfect information, where each consumer knows each commodity perfectly without any uncertainty; (ii) each consumer is a price taker; and (iii) prices are linear without quantity discount.

Let X_i ($\subseteq \mathbb{R}^\ell$) be the set of all consumptions possible for consumer i, known as his consumption set or his demand. Then, each consumption $x_i \in X_i$ generally contains a relatively small number of nonzero components, where the typical inputs consist of dated and location-specific products, goods and services, and the outputs various labors that are dated and located differently. That is, products, goods, services, and labor that become available and/or are delivered at different times and/or different locations are treated as different commodities. Without loss of generality, the following axioms are assumed throughout this presentation:

- (Lower Boundedness): For each i $(= 1,2,\dots,m)$, the consumption set X_i has a lower bound for the order relation $\leq$ defined on $\mathbb{R}^\ell$.
- (Comparability). If $x_i^1, x_i^2 \in X_i$ are comparable in terms of the preference of consumer i, as determined by his system of values and beliefs, then one and only one of the following alternatives holds true: (i) x_i^1 is preferred to x_i^2, written as $x_i^1 \succsim_i x_i^2$; (ii) x_i^1 is indifferent to x_i^2, written $x_i^1 \sim_i x_i^2$; and (iii) x_i^2 is preferred to x_i^1, written $x_i^1 \precsim_i x_i^2$.
- (Insatiability of preferences). For any $x_i \in X_i$, there is another $y_i \in X_i$ such that consumer i prefers y_i to x_i, written $x_i \prec_i y_i$.

While the second and third axioms seem obvious in real life, the first is justified as follows: The consumed quantity of each commodity has to be greater than or equal to zero; and the output of any commodity provided by a consumer must be bounded from above.

Other than using his system of values and beliefs to determine his consumption preferences, a consumer also employs this system to order real numbers in a particular way. For instance, an income in the amount of \$30K is mostly seen as less than that of \$3 million. However, when value-and-belief systems are involved, such orders as \$30K < \$3 million can be easily reversed. For example, assume that the former income is produced out of hard work, while the latter is the outcome of robbing a bank. Then, people with certain types of value-and-belief systems will easily order \$30K as greater than \$3 million. If we let $\leq_i$ (respectively, $<_i$, $>_i$, $\geq_i$, $=_i$) be consumer i's particular order of real numbers, there then are three order relations $\leq$ (respectively, $<$, $>$, $\geq$, $=$), $\precsim_i$ (respectively, $\prec_i$, $\succ_i$, $\succsim_i$, $=_i$) (defined on X_i) and $\leq_i$ (defined on $\mathbb{R}$). One needs to note that different from both $\leq$ and $\leq_i$, consumer i's consumption preferences $\precsim_i$ in real life are generally influenceable and often influenced by peers. Consumptive preferences may also be frequently altered temporarily by peer pressures, especially for emerging adults, as documented in Hu *et al.* (2021) and Mani *et al.* (2013). Because the concept of time is ignored in this paper, or what is considered here represents only a freezing moment in time, the relation $\precsim_i$ becomes fixed and not influenceable by peers.

The preference relation $\precsim_i$ is said to be a preorder if it satisfies (i) the property of reflexivity: for any $x_i \in X_i$, $x_i \precsim_i x_i$; and (ii) the property of transitivity: for any $x_i^1, x_i^2, x_i^3 \in X_i$, $x_i^1 \precsim_i x_i^2$ and $x_i^2 \precsim_i x_i^3$ imply $x_i^1 \precsim_i x_i^3$. It is said to be complete, if each pair $x_i^1, x_i^2 \in X_i$ can be compared by $\precsim_i$.

With the preceding background, we are now ready to present the main conclusions of our research.

4. Various Convex Preferences

Previous literature has identified three types of convex preferences – weak convexity, convexity and strong convexity. Debreu (1959) and following authors (e.g., Mas-Collel et al., 1995) have investigated these types in depth, and we now explore the properties of these three types of convex preferences in subsections in this part of the paper. Subsection 4.1 looks at the concept of weak convexity and the possibility that when the preference relation is not complete, a weighted combination of two possible consumptions may not be comparable with any of the consumptions in terms of the preference relation. Subsection 4.2 examines the concept of convex preferences and that concept's relationship with weak convexity. Subsection 4.3 looks at asymptotically preserving preferences. Subsection 4.4 studies those preference relations that satisfy the conditions of either additive conservation or positive multiplicativity. Subsection 4.5 investigates the concept of strong convexity. Finally, in Subsection 4.6, attention is turned to the development of these three types of convex preferences in general convex spaces.

4.1. Weakly Convex Preferences

Assume that in this section X_i is convex, that is, for any x^1 and $x^2 \in X_i$ and any scalar $\alpha \in [0,1]$, $\alpha x^1 + (1-\alpha)yx^2 \in X_i$. If X_i additionally satisfies that for distinct $x_i^1, x_i^2 \in X_i$ and arbitrary $\alpha \in (0,1)$,

$$x_i^1 \precsim_i x_i^2 \to x_i^1 \precsim_i \alpha x_i^2 + (1-\alpha)x_i^1, \tag{1}$$

then X_i is said to be weakly convex with respect to $\precsim_i$ (Debreu, 1959, p. 59). When no confusion is caused, the phrase "with respect to $\precsim_i$" will be omitted.

Proposition 1. If X_i is convex and the preference relation $\precsim_i$ is complete, then the set X_i of all possible consumptions is weakly convex if and only if

$$\forall\, x_i' \in X_i, \{x_i \in X_i: x_i' \precsim_i x_i\} \text{ is convex.} \tag{2}$$

Proof. (→). For any $x_i' \in X_i$, let $x_i^1, x_i^2 \in Z = \{x_i \in X_i: x_i' \precsim_i x_i\}$ and $\alpha \in (0,1)$. Without loss of generality, assume $x_i^1 \precsim_i x_i^2$. We need to show $\alpha x_i^2 + (1-\alpha)x_i^1 \in Z$.

To this end, we have $x_i' = \alpha x_i' + (1-\alpha)x_i' \precsim_i x_i^1 = \alpha x_i^1 + (1-\alpha)x_i^1$. Since $\precsim_i$ is assumed to be complete, x_i^1 and $\alpha x_i^2 + (1-\alpha)x_i^1$ are comparable with respect to $\precsim_i$. And,

$$x_i' \precsim_i x_i^1 = \alpha x_i^1 + (1-\alpha)x_i^1 \precsim_i \alpha x_i^2 + (1-\alpha)x_i^1. \tag{3}$$

Hence, $\alpha x_i^2 + (1-\alpha)x_i^1 \in Z$. That is, Z is convex.
(←). For any distinct $x_i^1, x_i^2 \in X_i$ and arbitrary $\alpha \in (0,1)$, satisfying $x_i^1 \precsim_i x_i^2$, we want to show $x_i^1 \precsim_i \alpha x_i^2 + (1-\alpha)x_i^1$ by assuming equation (2). Because $Z = \{x_i \in X_i: x_i^1 \precsim_i x_i\}$ is convex, from $x_i^2 \in Z$, it follows that $\alpha x_i^2 + (1-\alpha)x_i^1 \in Z$, for any $\alpha \in (0,1)$. That is, $x_i^1 \precsim_i \alpha x_i^2 + (1-\alpha)x_i^1$. So, equation (1) holds true. QED
The following example shows that if the preference relation $\precsim_i$ is not complete, there might be $x_i^1, x_i^2 \in X_i$ and a scalar $\alpha \in (0,1)$ such that x_i^1 and $\alpha x_i^2 + (1-\alpha)x_i^1$ are not comparable with respect to the preference relation $\precsim_i$ on X_i.

Example 1. Assume that the consumer i's system of values and beliefs defines the order relation $\leq_i$ of real numbers by referring to the mod4 function and the arc length between the comparing quantities. In particular, for two demanded quantities x_{ih}^1 and x_{ih}^2 of commodity h, $\leq_i = \leq_{mod(4)}$ such that $x_{ih}^1 <_{mod(4)} x_{ih}^2$ if and only if on the shorter arc between $x_{ih}^1\ mod(4)$ and $x_{ih}^2\ mod(4)$, the arrow points from $x_{ih}^1\ mod(4)$ to $x_{ih}^2\ mod(4)$, Figure 1.

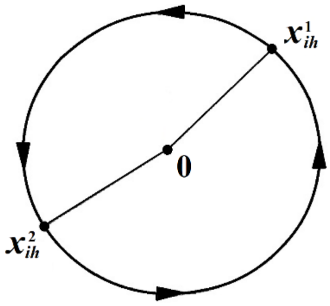

Figure 1 How the particular order $\leq_{\mathbf{mod(4)}}$ is defined
Source: self-research.

Define the preference relation $\precsim_i = \precsim_{mod(4)}$ as follows: For any $x_i^1, x_i^2 \in X_i \subseteq \mathbb{R}^\ell$,

$$x_i^1 \precsim_{mod(4)} x_i^2 \text{ if and only if } \forall\, h \left(x_{ih}^1 \leq_{mod(4)} x_{ih}^2\right).$$

Let $x_i^1, x_i^2 \in X_i$ be two consumptions such that

$$x_{ik}^1 = x_{ik}^2, k = 1, 2, \dots, \ell, k \neq h,$$

and

$$x_{ih}^1 = 1, \text{and } x_{ih}^2 = 3.5.$$

Then, for $\alpha = 2/2.5 \in (0,1)$, x_i^1 and $\alpha x_i^2 + (1-\alpha)x_i^1$ are not comparable with respect to $\precsim_{mod(4)}$. In particular, because $x_{ik}^1 = \alpha x_{ik}^2 + (1-\alpha)x_{ik}^1$, for $k = 1, 2, \dots, \ell, k \neq h$. The comparability of x_i^1 and $\alpha x_i^2 + (1-\alpha)x_i^1$ with respect to $\precsim_{mod(4)}$ becomes that of x_{ih}^1 and $\alpha x_{ih}^2 + (1-\alpha)x_{ih}^1$ with respect to $\leq_{mod(4)}$. And since $x_{ih}^1 = 1$ and $\alpha x_{ih}^2 + (1-\alpha)x_{ih}^1 = \left(\frac{2}{2.5}\right) \cdot 3.5 + \left(\frac{0.5}{2.5}\right) \cdot 1 = 3$, it follows that x_{ih}^1 and $\alpha x_{ih}^2 + (1-\alpha)x_{ih}^1$ are not comparable with respect to $\leq_{mod(4)}$. Hence, x_i^1 and $\alpha x_i^2 + (1-\alpha)x_i^1$ are not comparable with respect to $\precsim_{mod(4)}$.

Not only so, but it can also be seen that $x_i^2 \precsim_i x_i^1$, because $x_{ih}^2 = 3.5 \leq_{mod(4)} x_{ih}^1 = 1$. Therefore, this example shows that Proposition does not hold true, if the preference relation $\precsim_i$ is not complete. QED

Naturally, one question about this example arises. When does a mod function (also known as a modular operation) appear in real life? The answer is that we frequently see such operations in situations like 12-hour clocks, 7-day weeks, monthly cycles of various numbers of days, and durations of projects that follow one after another. In each of these cases, when a cycle is fully traversed, a new round of counting or measurement begins again from the starting mark 0. In terms of foods, a person's consumption of any particular food is not the more the better. In fact, the opposite is generally true: The one-time consumption of any food has to be limited within a certain upper bound. Otherwise, the food will become something the consumer avoids completely in the future!

In terms of the literature, the concept of weakly convex preferences is the same as that of convex preferences given in Mas-Collel et al. (1995, p. 44). Accordingly, Example 1 above shows the importance of this work: To make the consequent economic theory more relevant to real life, we must revisit the prevalent consumer theory. Through this revisit, we will see which previously developed conclusions are really true and what those conclusions will look like when preferences are not complete and transitive or when preferences are not presently known as rational (Mas-Collel et al., 1995; Miller, 2006).

4.2. Convex Preferences

Slightly different from the concept of weak convexity, if X_i is convex and the following condition holds true, then X_i is said to be convex with respect to $\precsim_i$ (Debreu, 1959, p. 60). For any distinct consumptions $x_i^1, x_i^2 \in X_i$ and arbitrary scalar $\alpha \in (0,1)$,

$$x_i^1 \prec_i x_i^2 \to x_i^1 \prec_i \alpha x_i^2 + (1-\alpha)x_i^1. \quad (4)$$

As before, when no confusion is caused, the phrase "with respect to $\precsim_i$" will be omitted.

Proposition 2. If X_i is convex and the preference relation $\precsim_i$ on X_i is a complete preorder, then X_i is convex with respect to $\precsim_i$ if and only if

$$\forall\, x_i' \in X_i, \{x_i \in X_i : x_i' \prec_i x_i\} \text{ is convex.} \quad (5)$$

Proof. (→) To show equation (5), for any $x_i' \in X_i$, let $Z = \{x_i \in X_i : x_i' \prec_i x_i\}$. Pick arbitrary $x_i^1, x_i^2 \in Z$ and let $\alpha \in (0,1)$ be an arbitrary scalar. Without loss of generality, assume that $x_i^1 \precsim_i x_i^2$. Then we have

$$x_i' \prec_i x_i^1 = \alpha x_i^1 + (1-\alpha)x_i^1.$$

Since $\precsim_i$ is assumed to be complete, x_i^1 and $\alpha x_i^2 + (1-\alpha)x_i^1$ are comparable. So, from $x_i^1 \precsim_i x_i^2$, it follows that

$$x_i' \prec_i x_i^1 = \alpha x_i^1 + (1-\alpha)x_i^1 \precsim \alpha x_i^2 + (1-\alpha)x_i^1.$$

So, $\alpha x_i^2 + (1-\alpha)x_i^1 \in Z$. Hence, Z is convex.
(←) Assume that for any $x_i' \in X_i$, $\{x_i \in X_i : x_i' \prec_i x_i\}$ is convex. Pick two arbitrary consumptions $x_i^1, x_i^2 \in X_i$ and an arbitrary scalar $\alpha \in (0,1)$, satisfying $x_i^1 \prec_i x_i^2$. Then the convexity of $Z = \left\{x_i \in X_i : x_i^1 \prec_i x_i\right\}$ implies that

$$x_i^1 = \alpha x_i^1 + (1-\alpha)x_i^1 \prec_i \alpha x_i^2 + (1-\alpha)x_i^1,$$

where the comparability of x_i^1 and $\alpha x_i^2 + (1-\alpha)x_i^1$ comes from the assumption that $\precsim_i$ is complete. QED

In terms of the literature, the concept of convex preferences, as seen above, is the same as that of strict convex preferences given in Mas-Collel et al. (1995, p. 44). And, Example 1 shows why the condition that the completeness of the preference preorder $\precsim_i$ is necessary for Proposition 2 to hold true.

For any consumption $x_i \in X_i$, define the indifference class (or curve) of x_i as follows (Debreu, 1959; Miller, 2006):

$$[x_i] = \{x_i' \in X_i : x_i' \precsim_i x_i \text{ and } x_i \precsim_i x_i' \}.$$

Let X_i^* be a subset of X_i, known as a set of (consumer i's) preference representations, such that for any x_i^1 and $x_i^2 \in X_i^*$, $x_i^1 \neq x_i^2$ implies $[x_i^1] \neq [x_i^2]$ and $X_i = \cup_{x_i \in X_i^*}[x_i]$. If a subset $X_i' \subseteq X_i^*$ satisfies that any consumptions x_i^1, $x_i^2 \in X_i'$ are comparable in terms of the preference relation $\precsim_i$, then X_i' is known as a chain in X_i^*. A chain X_i' in X_i^* is referred to as a maximal chain, provided that for any $x_i \in X_i$, if x_i is comparable with each element in X_i', then $x_i \in X_i'$. For more details on ordered sets, please consult with Kuratowski and Mostowski (1976).

Define a function $u_i: X_i \to X_i^*$ as follows: for any $x_i \in X_i$, $u_i(x_i) = x_i^* \in X_i^*$, if $x_i \in [x_i^*]$. Then, we treat u_i as the canonical utility function of consumer i, satisfying that for any x_i^1, $x_i^2 \in X_i$, $x_i^1 \precsim_i x_i^2$ if and only if $u_i(x_i^1) \precsim_i u_i(x_i^2)$. For any chosen maximal chain X_i^{max} in X_i^*, the preimage of u_i of the chain X_i^{max} is equal to

$$u_i^{-1}(X_i^{max}) = \cup \{[x_i^*]: x_i^* \in X_i^{max}\}.$$

The preference relation $\precsim_i$ is said to be continuous, if for any maximal chain X_i^{max} in X_i^* and any $x_i' \in u_i^{-1}(X_i^{max})$, the following sets are closed in $u_i^{-1}(X_i^{max})$:

$$\{x_i \in u_i^{-1}(X_i^{max}): x_i \precsim_i x_i'\} \text{ and } \{x_i \in u_i^{-1}(X_i^{max}): x_i \succsim_i x_i'\}. \quad (6)$$

In other words, the closedness of the first set in equation (6) means that for any sequence $\{x_i^q\}_{q=1}^{\infty}$ of consumptions possible for consumer i, if each x_i^q is at most as desired as x_i' and $x_i^q \to x_i^*$ (a consumption in X_i), then x_i^* is also at most as desired as x_i'. And similarly, the closedness of the second set in equation (6) is defined.

Proposition 3. If the preference relation $\precsim_i$ on X_i is a complete and continuous preorder, then X_i's convexity with respect to $\precsim_i$ implies X_i's weak convexity with respect to $\precsim_i$.

Proof. For any consumptions $x_i^1, x_i^2 \in X_i$ and any scalar $\alpha \in (0,1)$, assume that $x_i^1 \precsim_i x_i^2$. We want to show equation (1). To this end, let

$$[x_i^1, \rightarrow) = \{x_i \in X_i : x_i^1 \precsim_i x_i\}.$$

Since X_i is assumed to be convex, for any scalar $\alpha \in (0,1)$, $\alpha x_i^2 + (1-\alpha)x_i^1 \in X_i$. Because $\precsim_i$ is complete, it means that x_i^1 and $\alpha x_i^2 + (1-\alpha)x_i^1$ are comparable. The continuity of the preference relation $\precsim_i$ implies that $[x_i^1, \rightarrow)$ is closed in X_i. We need to prove $\alpha x_i^2 + (1-\alpha)x_i^1 \in [x_i^1, \rightarrow)$. By contradiction, assume the opposite holds true. That is, we assume that

$$\alpha x_i^2 + (1-\alpha)x_i^1 \in Z = \{x_i \in [x_i^1, \rightarrow) : x_i^1 \succ_i x_i\}. \quad (7)$$

In the rest of this proof, we show that Z is in fact an empty set. Assume that $Z \neq \emptyset$. Let us pick an element $x_i' \in Z$. From the fact that $[x_i^1, \rightarrow)$ is closed in X_i and that $(\leftarrow, x_i'] = \{x_i \in X_i : x_i \precsim_i x_i'\}$ is also close (since $\precsim_i$ is continuous), satisfying that $(\leftarrow, x_i'] \cap [x_i^1, \rightarrow) = \emptyset$, the assumption that X_i is convex implies that there is another point $x_i'' \in X_i$ such that $x_i' \prec_i x_i'' \prec_i x_i^1$. Hence, the convexity of X_i implies that for any scalar $\alpha \in (0,1)$

$$\begin{aligned} x_i'' &\prec_i \alpha x_i^1 + (1-\alpha)x_i'' \\ &\prec_i \alpha x_i' + (1-\alpha)x_i' \\ &= x_i', \end{aligned}$$

where the second line is from the fact that $x_i' \in Z$ and so $x_i' \in [x_i^1, \rightarrow)$. That contradicts with $x_i' \prec_i x_i''$. That is, $Z = \emptyset$ so that equation (7) is impossible. That is, $\alpha x_i^2 + (1-\alpha)x_i^1 \in [x_i^1, \rightarrow)$, which confirms that X_i is weakly convex. QED

4.3. Asymptotically Preserving Preferences

If for each sequence $\{x_i^q\}_{q=1}^{\infty}$ of consumptions in X_i, satisfying $x_i^q \succsim_i x_i^0$ (respectively, $x_i^q \precsim_i x_i^0$), for every q and some $x_i^0 \in X_i$, $\lim_{q\rightarrow\infty} x_i^q \succsim_i x_i^0$ (respectively, $\lim_{q\rightarrow\infty} x_i^q \precsim_i x_i^0$), whenever the limit exists, then we say that consumer i's consumptions asymptotically preserve preference preordering $\precsim_i$.

Proposition 4. If consumer i's preferable consumptions asymptotically preserve preference preordering $\precsim_i$ and X_i is convex with respect to $\precsim_i$, then for any $x_i' \in X_i$, x_i' is a limit point of $\{x_i \in X_i : x_i \succ_i x_i'\}$.

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Proof. The convexity of X_i with respect to $\precsim_i$ implies that for any $z_i \in \{x_i \in X_i : x_i \succ_i x'_i\}$,

$$x'_i \prec_i \alpha z_i + (1-\alpha)x'_i, \text{for any } \alpha \in (0,1), \quad (8)$$

where the existence of consumption z_i comes from the assumption that no satiation consumption exists for consumer i.

Therefore, for $q = 1/n$, $n = 2,3,...$, the sequence $\{x_i^q\}_{q=1}^{\infty}$, where $x_i^q = qz_i + (1-q)x'_i$, converges to x'_i. So, equation (8) indicates that x'_i is a limit point of $\{x_i \in X_i \colon x_i \succ_i x'_i\}$. QED

An indifferent class is said to be thick (Debreu, 1959) if its interior in X_i is not empty. What Proposition 4 says is that the indifference class of each consumption of consumer i is not thick, when his set X_i of possible consumptions is convex with respect to $\precsim_i$ and asymptotically preserves preference preordering.

The following example demonstrates that, generally, not every convergent sequence of preferable consumptions of a consumer asymptotically preserves his preference preordering.

Example 2: Assume that the system of values and beliefs of consumer i ranks the quantity of a specific commodity h as follows: for any two real numbers x and y, $x <_i y$ if and only if $x(\text{mod}4) < y(\text{mod}4)$. Let $x_i^1, x_i^2, ..., x_i^q, ... \in X_i$ be a sequence of possible consumptions for consumer i such that

$$x_{ik}^q = x_{ik}^1, k = 1, 2, ..., \ell, k \neq h, q = 1,2, ... \quad (9)$$

and

$$x_{ih}^q = 3 + \frac{q}{q+1}, q = 1,2, ... \quad (10)$$

Then, it is ready to see that $x_i^q \to x_i^0$, as $q \to \infty$, where $x_{ik}^0 = x_{ik}^1, k = 1, 2, ..., \ell, k \neq h$, and $x_{ih}^0 = 0$, which is equal to 4 $(mod4)$.

Define x_i^{low} as follows: $x_{ik}^{low} = x_{ik}^1, k = 1, 2, ..., \ell, k \neq h$, and $x_{ih}^{low} = 3$. Then, equations (9) and (10) imply that

$$x_i^q \succsim_i x_i^{low} \text{ and } \lim_{q\to\infty} x_i^q = x_i^0 \prec_i x_i^{low}. \text{QED}$$

The significance of Example 2 is that it confirms the fact that $\{x_i \in X_i\colon x_i \succ_i x_i'\}$ is generally different from $\{x_i \in X_i\colon x_i \succsim_i x_i'\}$. That disproves an equivalence relation in Debreu (1959, p. 59).

4.4. Additively Conserved and Positively Multiplicative Preferences

A preference relation $\precsim_i$ on X_i is said to satisfy the condition of additive conservation, if for any consumptions $a_i^j, b_i^j \in X_i$, $j = 1,2$,

$$a_i^1 \precsim_i b_i^1 \text{ and } a_i^2 \precsim_i b_i^2 \ \rightarrow a_i^1 + a_i^2 \precsim_i b_i^1 + b_i^2, \tag{11}$$

where the sign $\precsim_i$ will become $\prec_i$ in the consequence if $\prec_i$ appears in at least one of the two antecedents.

The preference relation $\precsim_i$ is said to satisfy the condition of positive multiplicativity, if for any consumptions $x_i^1, x_i^2 \in X_i$ and any scalar $\alpha > 0$,

$$x_i^1 \precsim_i x_i^2 \ \rightarrow \alpha x_i^1 \precsim_i \alpha x_i^2,$$

where the sign $\precsim_i$ will become $\prec_i$ in the consequence if $\prec_i$ appears in the antecedent.

Proposition 5. If the preference relation $\precsim_i$ on a connected X_i satisfies the conditions of both positive multiplicativity and additive conservation, then X_i is both weakly convex and convex with respect to $\precsim_i$.

Proof. For any consumptions $x_i^1, x_i^2 \in X_i$ and arbitrary scalar $\alpha \in (0,1)$, if $x_i^1 \precsim_i x_i^2$, then the condition of positive multiplicativity implies

$$\alpha x_i^1 \precsim_i \alpha x_i^2 \text{ and } (1-\alpha)x_i^1 \precsim_i (1-\alpha)x_i^1.$$

Now, the condition of additive conservation guarantees that

$$x_i^1 = \alpha x_i^1 + (1-\alpha)x_i^1 \precsim_i \alpha x_i^2 + (1-\alpha)x_i^1.$$

Since X_i is connected, we have $\alpha x_i^2 + (1-\alpha)x_i^1 \in X_i$. This is, X_i is weakly convex. And similarly, we can show the convexity of X_i. QED

The following example shows that not all preorders satisfy the condition of additive conservation.

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Example 3: Assume that consumer i's system of values and beliefs preorders the quantities of a particular commodity h by referring to the mod4 function so that for any two real numbers x and y, $x \prec_i y$ if and only if $x(\text{mod}4) < y(\text{mod}4)$. Let $x_i^1, x_i^2, x_i^3 \in X_i$ be three consumptions such that

$$x_{ik}^1 = x_{ik}^2 = x_{ik}^3, k = 1, 2, \dots, \ell, k \neq h,$$

and

$$x_{ih}^1 = 2, x_{ih}^2 = 3 \text{ and } x_{ih}^2 = 1.$$

Then, we have $x_i^1 \precsim_i x_i^2$ and $x_i^3 \precsim_i x_i^3$. However, instead of $x_i^1 + x_i^3 \precsim_i x_i^2 + x_i^3$, we have

$$x_i^1 + x_i^3 \succsim_i x_i^2 + x_i^3,$$

because

$$x_{ik}^1 + x_{ik}^3 = x_{ik}^2 + x_{ik}^3, k = 1, 2, \dots, \ell, k \neq h,$$

and

$$x_{ih}^1 + x_{ih}^3 = 3 \succsim_i x_{ih}^2 + x_{ih}^3 = 3 + 1 \ =_{mod4} 0.$$

That is, Proposition 5 is not guaranteed to be generally true without assuming that the preference relation $\precsim_i$ on X_i satisfies the condition of additive conservation. Specifically, the implication in equation (11) cannot be successfully guaranteed without assuming additive conservation. QED
Similar to the previous example, the following one demonstrates that not all preference relations satisfy the condition of positive multiplicativity.

Example 4. Continuing from Example 3, it is ready to see that the preference relation $\precsim_i$ (= $\leq_{mod(4)}$) of commodity h does not satisfy the condition of positive multiplicativity. In fact, for how positive multiplicativity is violated, we have

$$1 \precsim_i \left(\text{or} \leq_{mod(4)}\right) 2 \ \nrightarrow 2 \cdot 1 \precsim_i \left(\text{or} \leq_{mod(4)}\right) 2 \cdot 2$$

where the left-hand side is actually $2 \cdot 1 = 2 \succsim_i \left(\text{or } \geq_{mod(4)}\right) 2 \cdot 2 = 0$ = the right-hand side. In other words, let $x_i^1, x_i^2 \in X_i$ be two consumptions such that

$$x_{ik}^1 = x_{ik}^2, k = 1, 2, \dots, \ell, k \neq h,$$

and

$$x_{ih}^1 = 1, x_{ih}^2 = 2.$$

Then we have $x_i^1 \precsim_i x_i^2$ and $2x_i^1 \succsim_i 2x_i^2$. That is, the condition of positive multiplicativity does not hold true for consumer i's $\precsim_i$. QED

4.5. Strongly Convex Preferences

If for any two indifferent consumptions x_i^1 and x_i^2, their weighted average with an arbitrary positive weight are preferred to both of them, then X_i is said to be strongly convex with respect to $\precsim_i$ (Debreu, 1959, p. 61). Symbolically, X_i is strongly convex with respect to $\precsim_i$, if for any consumptions $x_i^1, x_i^2 \in X_i$ and arbitrary scalar $\alpha \in (0,1)$,

$$x_i^1 \sim_i x_i^2 \rightarrow x_i^1 \prec_i \alpha x_i^2 + (1-\alpha)x_i^1. \quad (12)$$

As before, when no confusion is caused, the phrase "with respect to $\precsim_i$" will be omitted.

Proposition 6. Assume that each infinity can be actually (not potentially) achieved. If for each maximal chain $X_i^{max} \subseteq X_i$, both $u_i^{-1}(X_i^{max})$ is a connected subset of $\mathbb{R}^\ell$ and X_i is convex, where $u_i: X_i \rightarrow X_i^*$ is the canonical utility function, and the preference relation $\precsim_i$ is a continuous preorder, then X_i's strong convexity with respect to $\precsim_i$ implies X_i's convexity with respect to $\precsim_i$.
Proof. Assume that X_i is strongly convex with respect to $\precsim_i$ so that for any consumptions $x_i^1, x_i^2 \in X_i$ and arbitrary scalar $\alpha \in (0,1)$, equation (12) holds true, where the conclusion $\alpha x_i^2 + (1-\alpha)x_i^1 \in X_i$ comes from the convexity of X_i. We need to show if these consumptions x_i^1, x_i^2 satisfy $x_i^1 \prec_i x_i^2$ then

$$x_i^1 \prec_i \alpha x_i^2 + (1-\alpha)x_i^1 \quad (13)$$

holds. To this end, let X_i^{max} be a maximal chain in X_i such that $u_i^{-1}(X_i^{max})$ contains both x_i^1 and x_i^2. Then the connectedness of $u_i^{-1}(X_i^{max})$ implies that $\alpha x_i^2 +$

$(1-\alpha)x_i^1 \in u_i^{-1}(X_i^{max})$. That is, $\alpha x_i^2 + (1-\alpha)x_i^1$ is comparable with each of x_i^1 and x_i^2.

Let u_i^r be a real-number valued utility function on $u_i^{-1}(X_i^{max})$. Under the assumption that each infinity can be actually (not potentially) achieved, the connectivity of $u_i^{-1}(X_i^{max})$ and the continuity of $\precsim_i$ jointly imply that this utility function u_i^r exists according to the famous Debreu's (1959) existence theorem. In particular, the original argument for the existence of the desired utility function u_i^r will go through in its entirety, except that both steps 1 and 2 (Debreu, 1959, p. 57-58) cannot be successfully completed without the assumption that each infinity can be actually (not potentially) achieved.

To understand this last statement, let us briefly examine the concept of infinities. This concept roughly deals with two kinds of infinities. One is known as potential infinities; and the other actual infinities (Forrest, 2013; Lin, 2008). Potential infinities denote processes that are forever, ongoing, and never-ending; and actual infinities denote processes that truly complete or had previously been completed. Relating to Debreu's (1959) existence theorem, its constructive process cannot be completed, unless one assumes that each infinity can be actually (not just potentially) achieved.

Since u_i^r is an increasing function on $u_i^{-1}(X_i^{max})$, we have $u_i^r(x_i^1) < u_i^r(x_i^2)$. And because for any scalar $\alpha \in (0,1)$, $x_i = \alpha x_i^2 + (1-\alpha)x_i^1$ is located between x_i^1 and x_i^2, we obtain $u_i^r(x_i^1) < u_i^r(x_i') < u_i^r(x_i^2)$. Therefore, from the definition of real-number valued utility functions, it follows that equation (13) holds true. That is, X_i is convex with respect to $\precsim_i$. QED

Proposition 7. If the following conditions hold true, then for any indifferent consumptions $x_i^1, x_i^2 \in X_i$, the weighted average $\alpha x_i^2 + (1-\alpha)x_i^1$ is also indifferent from x_i^1 and x_i^2, for any scalar $\alpha \in (0,1)$:

(i) Each infinity can be actually (not potentially) achieved.

(ii) For each maximal chain $X_i^{max} \subseteq X_i$, $u_i^{-1}(X_i^{max})$ is a connected subset of $\mathbb{R}^\ell$ and X_i are convex, where $u_i: X_i \to X_i^*$ is the canonical utility function,

(iii) The preference relation $\precsim_i$ on X_i is a complete and continuous preorder, and

(iv) X_i is strongly convex with respect to $\precsim_i$,

Proof. From Proposition 3, it follows that X_i is weakly convex. Hence, for any indifferent consumptions $x_i^1, x_i^2 \in X_i$, we have $x_i^1 \precsim_i x_i^2$ and $x_i^2 \precsim_i x_i^1$. So, the weak convexity of X_i implies that $x_i^1, x_i^2 \precsim_i \alpha x_i^2 + (1-\alpha)x_i^1$, for any scalar $\alpha \in (0,1)$. Then a similar argument as the one in the proof of Proposition 6 produces $\alpha x_i^2 + (1-\alpha)x_i^1 \precsim_i x_i^1, x_i^2$. Therefore, the desired conclusion follows. QED

Proposition 8. If the set X_i of consumption possibilities is strongly convex and the preference relation $\precsim_i$ is a preorder, then for any $x_i \in X_i$ and any maximal chain $X_i^{max} \subseteq X_i$ such that $x_i \in u_i^{-1}(X_i^{max})$, where $u_i: X_i \to X_i^*$ is the canonical utility function, then the indifference class $[x_i]$ does not contain any non-degenerate closed segment of $u_i^{-1}(X_i^{max})$.

Proof. By contradiction, assume that the given conditions are satisfied while there are (i) a particular consumption $x_i \in X_i$, (ii) a maximal chain $X_i^{max} \subseteq X_i$ such that $u_i^{-1}(X_i^{max})$ contains x_i, and (iii) two distinct consumptions $x_i^1, x_i^2 \in u_i^{-1}(X_i^{max})$ such that the closed segment $[x_i^1, x_i^2] \subseteq [x_i] \subseteq u_i^{-1}(X_i^{max})$, where $[x_i]$ is the indifference class of x_i.

The assumption that $\precsim_i$ is a preorder implies that for any $z_i \in [x_i^1, x_i^2]$, $z_i \sim_i x_i$. However, the assumed strong convexity of X_i implies that $x_i^1 \prec_i z_i = \alpha x_i^2 + (1-\alpha)x_i^1$, for any $\alpha \in (0,1)$. This end contradicts the previous statement since $z_i = \alpha x_i^2 + (1-\alpha)x_i^1$ belongs to the segment $[x_i^1, x_i^2]$. That is, for any $x_i \in X_i$ and any maximal chain X_i^{max} such that $u_i^{-1}(X_i^{max})$ contains x_i, the indifference class $[x_i]$ does not contain any non-degenerate closed segment of $u_i^{-1}(X_i^{max})$. QED

4.6. Abstract Convex Structures

The technique used to establish the previous model of consumption sets is to employ the quantities of commodities that a consumer desires to consume. Hence, it is natural for us to think about how to directly examine the entire collection of various bundles of commodities without first identifying that collection with a subset in the Euclidean space $\mathbb{R}^\ell$. To this end, abstract convex structures come to mind.

Let X be a non-empty set and $\mathbb{C}$ a family of subsets of X. The family $\mathbb{C}$ is known as a convexity on X (Kubis, 1999; van de Vel, 1993) if the following conditions hold true:

(a) $\emptyset, X \in \mathbb{C}$;

(b) For any sub-family $\mathbb{C}' \subseteq \mathbb{C}$, $\bigcap\{Z: Z \in \mathbb{C}'\} \in \mathbb{C}$ (or $\mathbb{C}$ is closed under arbitrary intersections);

(c) For any sub-family $\{Z_i: i \in I\} \subseteq \mathbb{C}$, where I is an index set if $\{Z_i: i \in I\}$ is a chain in terms of the inclusion relation $\subseteq$ between sets, then $\bigcup\{Z_i: i \in I\} \in \mathbb{C}$ (or $\mathbb{C}$ is closed under unions of chains).

In this case, each element in $\mathbb{C}$ is known as a convex (sub)set of X; and the ordered pair $(X, \mathbb{C})$ a convex space. For $Z \in \mathbb{C}$, if its complement $Z^C = X - Z$ also belongs to $\mathbb{C}$, then Z is known as a half-space. Let $\mathcal{H}$ denote the set of all half-spaces in $\mathbb{C}$. Given a convex space $(X, \mathbb{C})$ and a subset $A \subseteq X$, the convex hull of A, denoted by $co(A)$, is defined as follows: $co(A) = \cap\{Z \in \mathbb{C}: A \subseteq Z\}$. It can be seen readily that $Z \in \mathbb{C}$ if and only if $co(Z) = Z$. For n ($\in \mathbb{N}$) many arbitrary elements $x^i \in \mathbb{C}$, $i =$

$1,2,\dots,n$, where $\mathbb{N}$ is the set of all natural numbers, $co\left(\{x^i\}_{i=1}^n\right)$ is known as the n-polytope of the elements, while the 2-polytope $co(\{x^1,x^2\}) = co(x^1,x^2)$ is known as the segment joining x^1 and x^2.
A convex space $(X,\mathbb{C})$ (or convexity $\mathbb{C}$) is known as N-ary, for some fixed $N \in \mathbb{N}$, if for any $A \subseteq X$, $co(F) \subseteq A$, whenever $F \subseteq A$ contains at most N elements, then $A \in \mathbb{C}$. A 2-ary convexity is also known as an interval convexity.

Example 5. Show that the ordered pair $(\mathbb{R}^\ell,\mathbb{C})$, where $\mathbb{C}$ is the collection of all convex subsets in $\mathbb{R}^\ell$, is an interval (or 2-ary) convex space.
To this end, it is ready to see that conditions (a) and (b) are satisfied. To show condition (c), let $\{Z_i : i \in I\} \subseteq \mathbb{C}$ be an arbitrary chain in terms of $\subseteq$, $x^1, x^2 \in \bigcup\{Z_i : i \in I\}$ two arbitrary elements, and $\alpha \in (0,1)$ an arbitrary scalar. There then are $i_1, i_2 \in I$ such that $x^1 \in Z_{i_1}$ and $x^2 \in Z_{i_2}$. Because $\{Z_i : i \in I\}$ is a chain in terms of $\subseteq$, it follows that either $Z_{i_1} \subseteq Z_{i_2}$ or $Z_{i_2} \subseteq Z_{i_1}$. That is, we have either $x^1, x^2 \in Z_{i_2}$ or $x^1, x^2 \in Z_{i_1}$. No matter which is the case, it follows that $\alpha x^1 + (1-\alpha)x^2 \in Z_{i_1}$ or $Z_{i_2} \subseteq \bigcup\{Z_i : i \in I\}$. Therefore, $\bigcup\{Z_i : i \in I\} \in \mathbb{C}$.
To show that $(\mathbb{R}^\ell,\mathbb{C})$ is 2-ary convex space, let $A \subseteq \mathbb{R}^\ell$ be a subset and $x^1, x^2 \in A$ be arbitrary such that $co(x^1,x^2) \subseteq A$, we need to show that A is a convex subset. That is equivalent to demonstrating that $co(x^1,x^2) = \{\alpha x^1 + (1-\alpha)x^2 : \alpha \in [0,1]\}$. To this end, let $z^1, z^2 \in co(x^1,x^2)$ and $\alpha \in (0,1)$ be arbitrary. So, there are $\alpha^1, \alpha^2 \in [0,1]$ such that

$$z^1 = \alpha^1 x^1 + (1-\alpha^1)x^2 \text{ and } z^2 = \alpha^2 x^1 + (1-\alpha^2)x^2.$$

Therefore,

$$\alpha z^1 + (1-\alpha)z^2 = \beta x^1 + [1-\beta]x^2$$

where $\beta = \alpha\alpha^1 + (1-\alpha)\alpha^2 \in [\alpha^1,\alpha^2] \subseteq [0,1]$. That is, what is shown is that $\{\alpha x^1 + (1-\alpha)x^2 : \alpha \in [0,1]\}$ is convex so that $co(x^1,x^2) = \{\alpha x^1 + (1-\alpha)x^2 : \alpha \in [0,1]\}$. QED
For a convex space $(X,\mathbb{C})$, a preference relation $\precsim$ defined on X is said to be weak convex, if for any $x^1, x^2 \in X$, $x^1 \precsim x^2$ implies that for any $x \in co(\{x^1,x^2\})$, $x^1 \precsim x$. This concept extends that of weak convex preferences proposed in Subsection 4.1 for the convex space $(\mathbb{R}^\ell,\mathbb{C})$, where $\mathbb{C}$ is the collection of all convex sets in $\mathbb{R}^\ell$. Because of the following Proposition 9, we can see that in 2-ary convex space a weak convex preference, as defined here, is the same as convex preference, as so defined in Cardin (2019; 2022).

Proposition 9. If $(X, \mathbb{C})$ is a 2-ary convex space and $\precsim$ a complete preorder on X, then $\precsim$ is weak convex if and only if for every $z \in X$, the set $\{x \in X: z \precsim x\}$ is convex.

Evidently, this proposition generalizes Proposition 1 from the Euclidean space $\mathbb{R}^\ell$ to the case of any convex space $(X, \mathbb{C})$, where X can be any general set of elements.

Proof. ($\Rightarrow$) Let $x^1, x^2, x' \in X$ such that $x^1, x^2 \in \{x \in X: z \precsim x\}$ and $x' \in co(\{x^1, x^2\})$. If $x^1 \precsim x^2$, then from the assumption that $\precsim$ is weak convex, it follows that $z \precsim x^1 \precsim x'$. So, $x' \in \{x \in X: z \precsim x\}$. Since $(X, \mathbb{C})$ is a 2-ary convex space, we conclude that $\{x \in X: z \precsim x\}$ is convex.

($\Leftarrow$) Assume that for any $z \in X$, $\{x \in X: z \precsim x\}$ is convex. Hence, if $x^1, x^2 \in X$ satisfy $x^1 \precsim x^2$, then $x^1, x^2 \in \{x \in X: x^1 \precsim x\}$ and if $z \in co(\{x^1, x^2\})$, then $z \in \{x \in X: x^1 \precsim x\}$. QED

Proposition 10. Let $(X, \mathbb{C})$ be a convex space and $\precsim$ a complete preorder on X. Then, for every $z \in X$, the set $\{x \in X: z \precsim x\}$ is convex, if and only if for any $x^1, x^2, \ldots, x^n \in X$, for any $n \in \mathbb{N}$,

$$z \precsim x^i, \forall i, 1 \le i \le n, \rightarrow \forall x \in co(\{x^1, x^2, \ldots, x^n\}), z \precsim x. \quad (14)$$

Proof. ($\Rightarrow$) If $\{x \in X: z \precsim x\}$ is convex, then for each i, $1 \le i \le n$, $z \precsim x^i$ implies $co(\{x^1, x^2, \ldots, x^n\}) \subseteq \{x \in X: z \precsim x\}$.

($\Leftarrow$) If for every finite subset F of $Z = \{x \in X: z \precsim x\}$ we have $co(F) \subseteq Z$, then by Proposition 2.1 of De Vel (1993), we can prove that $co(Z) = Z$. Moreover, it is straightforward to show that if $\{x \in X: x^1 \precsim x\}$ is convex and $x^1 \precsim x^2$, then $co(\{x^1, x^2\}) \subseteq \{x \in X: x^1 \precsim x\}$. QED

Example 6. For a given convex space $(X, \mathbb{C})$, let us define a preference relation $\precsim$ on X as follows: for any $x, y \in X$, $y \precsim x$, if

$$\{H \in \mathcal{H}: x \in H\} \supseteq \{H \in \mathcal{H}: y \in H\}.$$

Then, $\precsim$ is reflexive, because for any $x \in X$, $\{H \in \mathcal{H}: x \in H\} \supseteq \{H \in \mathcal{H}: x \in H\}$. The relation $\precsim$ is transitive, because for any $x, y, z \in X$, if $y \precsim x$ and $z \precsim y$, then by definition we have

$$\{H \in \mathcal{H}: x \in H\} \supseteq \{H \in \mathcal{H}: y \in H\} \supseteq \{H \in \mathcal{H}: z \in H\}.$$

Evidently, for each half-space $H \in \mathcal{H}$, if $x \in H$ and $y \notin H$, then x and y are not comparable in terms of the afore-defined relation $\precsim$. That is, $\precsim$ is a preorder defined on X; but, it is not complete.
For any $z \in X$, it can be seen that

$$\{x \in X: z \precsim x\} = \cap \{H \in \mathcal{H}: z \in H\},$$

so that condition (b) of convex spaces implies that $\{x \in X: z \precsim x\}$ is convex. Hence, the preorder $\precsim$ is a weak convex preference. QED
By combining Propositions 9 and 10, it can be readily seen that when $\precsim$ satisfies the condition in equation (14), $\precsim$ is weak convex.
As for the concept of convex preferences, as proposed in Subsection 4.2, we can generalize it to the framework of abstract convex structures. In particular, for a convex space $(X, \mathbb{C})$, a preference relation $\precsim$ defined on X is said to be convex, if for any $x^1, x^2 \in X$, $x^1 \prec x^2$ implies that for each $x \in co(\{x^1, x^2\})$, if $x \nsim x^1$, then $x^1 \prec x$. For convex preferences, we have the following result:

Proposition 11. If $(X, \mathbb{C})$ is a 2-ary convex space and $\precsim$ a complete preorder, then $\precsim$ is convex if and only if for each $z \in X$, the set $\{x \in X: z \prec x\}$ is convex. QED
If a complete preference is considered, the following can also be shown.

Proposition 12. Let $\precsim$ be a complete preorder in a convex space $(X, \mathbb{C})$ such that for every $z \in X$, the set $\{x \in X: z \prec x\}$ is convex. Then, for each $z \in X$, the set $\{x \in X: z \precsim x\}$ is convex.
Proof. Since $\precsim$ is a complete preorder, we have

$$\{x \in X: z \precsim x\} = \bigcap_{y \prec z} \{x \in X: y \prec x\}$$

for every $z \in X$. Because each $\{x \in X: y \prec x\}$ belongs to $\mathbb{C}$, we can conclude that $\{x \in X: z \precsim x\}$ is a convex set. QED
Hence, as a corollary of Proposition 12, the below result readily follows.

Proposition 13. If $(X, \mathbb{C})$ is a 2-ary convex space and $\precsim$ a convex complete preorder, then $\precsim$ is weak convex. QED
Like the concepts of weakly convex preferences and convex preferences, the concept of strongly convex preferences can be defined for the general convex space $(X, \mathbb{C})$. Specifically, a preference relation $\precsim$ defined on the convex space $(X, \mathbb{C})$ is said to be strongly convex, if for any $x, y, z \in X$, $x \sim y$ and $B]x, z, y[$ imply $x \prec z$, where $B \subseteq$

X^3 is the betweenness relation defined (Cardin, 2019) as follows: $(x, z, y) \in B$ if and only if $\forall C \in \mathbb{C}, x, y \in C \rightarrow z \in C$. For convenience, $(x, z, y) \in B$ is written also as $B[x, z, y]$ and is understood as z is located between x and y with the possibility that $z \sim x$ or $z \sim y$. To emphasize the fact that z is between x and y but neither $z \sim x$ nor $z \sim y$, the symbol $B]x, z, y[$ is used.

5. Some Final Words

At the beginning of this paper, we posed two questions that were both theoretically and practically important. Those questions were: (1) under what conditions will those basic properties of consumer preferences that have been derived on the assumption of complete preferences still hold true when preferences are incomplete? (2) under an incomplete-preference scenario, what will the three convexities of consumer preferences – weak convexity, convexity, and strong convexity – look like? We have now answered those questions by employing the methods of Euclidean and general convex spaces. Taken together, the six examples derived throughout this paper have shown that

- To derive some of the most basic results of consumer theory, the preference relation of a consumer needs to be complete;
- Not all consumers have asymptotically preserving preferences;
- The conditions of additive conservation and positive multiplicativity do not generally hold true for the consumption preferences of each and every consumer;
- On a general convex space, a weakly convex preference preorder can be naturally induced by the collection of half-spaces.

By recognizing the multidimensionality of each consumer's physiological needs, we have removed the completeness assumption imposed on each consumer's set of consumption possibilities by most previous literature (e.g., Hervés-Beloso & Cruces, 2019; Mas-Collel et al., 1995). Doing so has enabled us to reintroduce the concepts of weak convexity, convexity, and strong convexity for consumer preferences and to reexamine the validity of some basic properties of those preferences. Other than confirming the necessity of a few key conditions in order for some known properties to hold, we are able to study issues not faced before so that brand-new results are developed.

Beyond its theoretical significance, this paper should therefore assist in developing a new, more practically applicable consumer theory that was so loudly called for by Paul Krugman (*New York Times*, 2009-09-02), Paul De Grauwe (Financial Times, 2009-07-21), and others.

Still, due to the length limitations of this paper, many important questions remain for future consideration. Two examples are: (1) under the condition that a consumer's set of consumption possibilities is not completely ordered by his preferences, what are the key properties of non-convex preferences (Halevy et al., 2017)? (2) riding on the various concepts of convex preferences noted in Subsection 4.6, how do we generalize Propositions 3 – 8 to the case of general convex spaces (Kubis, 1999; van de Vel, 1993)? We look forward to seeing these and other questions answered in the future.

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